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Vortex lattices as a key model for the fractional quantum Hall effect

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Abstract

It is demonstrated that all observed fractions at moderate Landau level fillings for the quantum Hall effect can be obtained without recourse to the phenomenological concept of composite fermions. The necessary additional flux is supplied by the vortex lattice, which allows us to consider all fractions in a unified frame. The group classification predicts the electron density of the ground state and the existence of a gap that separates it from excited states. This gap was calculated for some lattices in a simplified model.

The experimental discoveries of the integer quantum Hall effect (IQHE) by K von Klitzing (1980) and the fractional quantum Hall effect (FQHE) by Tsui, Stormer and Gossard (1982) were among the most outstanding achievements in condensed matter physics in the last century.

A qualitative theory of the IQHE can be derived for non-interacting electrons filling the lowest LL separated by an energy gap from the excited states in the presence of impurities (see reviews [1, 2]). But the FQHE with fractional LL fillings cannot be explained within the one-electron framework due to the macroscopic degeneracy of the LL, in contrast to the IQHE case. Various methods were used to overcome this difficulty. Laughlin proposed his famous variational many-electron wavefunction to explain the $1/3$ and other odd inverse fillings [3, 4]. The explanation of other observed fractions was obtained using various phenomenological hierarchical schemes, with the construction of 'daughter' states from the basic ones (Haldane 1983, Laughlin 1984, Halperin 1984).

In those works, the approximation of an extremely high magnetic field was used and all states were constructed from the states of the lowest Landau level. However, this does not conform to the experimental situation where the cyclotron energy is of the order of the mean energy of electron–electron interaction. Therefore, these theories are inevitably qualitative by nature. Moreover, this approach encounters difficulties when generalized to other fractions. Computer simulations also give a rather crude approximation of the realistic multiparticle functions since the number of particles in the corresponding computations on modern computers does not exceed several dozen.

The most successful phenomenological description is given by Jain's model of 'composite' fermions [5, 6], which predicts the majority of observed fractions. According to this model, electrons are dressed with magnetic flux quanta with the magnetic field concentrated in an infinitely narrow region in the vicinity of each electron. It is assumed that having an even number of flux quanta ensures that these particles are fermions. The inclusion of this additional magnetic field into the formalized theory leads to the so-called Chern–Simons Hamiltonian. This approach is described in detail in [7].

However, this theory gives an artificial six-fermion interaction whereas the actual calculations use the crude approximation of the constant 'effective' magnetic field as the sum of the external magnetic field and some additional artificial field that provides the total magnetic flux dictated by Jain's model of composite fermions. An additional difficulty in the calculations arises due to the fact that the effective electron mass (entering the expression for the energy gaps) is not known.

In the present work we show how to remove some restrictions of the Jain–Chern–Simons model related to the fillings outside Jain's series; we also derive the means to calculate the gap, i.e. to determine the 'effective' mass. The model used also does not change the nature of the standard interaction of electrons. The main concept is based on the topological classification of quantum states. There are quite a number of topological structures in condensed matter physics: vortex lattices in a rotating superfluid, Abrikosov vortices in superconductors, skyrmions in 2D electron systems at the lowest LL filling. It is difficult to give an exact topological classification of multiparticle wavefunctions for various physical systems. Probably the most simple and general definition can be obtained by using canonical transformation of the field operators of the second

quantization. A canonical transformation of the field operators is one that leaves the commutation relations intact. The Jain–Chern–Simons Hamiltonian with a modified e–e interaction was obtained via a rather complicated nonlinear canonical transformation [7]. The use of canonical transformations in condensed matter physics goes back to Bogolyubov's works on superfluidity (1947) and superconductivity (1957).

In analogy to the work [7], we consider the simplest case of the fermion canonical transformation not involving spin degrees of freedom and assume full polarization of 2D electrons:

$$\psi(\mathbf{r}) = \mathrm{e}^{\mathrm{i}\alpha(\mathbf{r})}\chi, \tag{1}$$

$$\psi^+(\mathbf{r}) = \chi^+\mathrm{e}^{-\mathrm{i}\alpha(\mathbf{r})}, \tag{2}$$

with $\alpha(\mathbf{r})$ having vortex-like singularities. It is evident that χ and χ^+ satisfy fermionic commutation relations if ψ and ψ^+ do. Inserting these expressions into the standard Hamiltonian for interacting electrons (spin indices omitted)

$$H = \frac{\hbar^2}{2m}\int \psi^+\left(-\mathrm{i}\nabla - \frac{e}{c\hbar}\mathbf{A}\right)^2\psi\,\mathrm{d}^2r + \int \frac{U(\mathbf{r}-\mathbf{r}')}{2}\psi^+(\mathbf{r})\psi^+(\mathbf{r}')\psi(\mathbf{r}')\psi(\mathbf{r})\,\mathrm{d}^2r\,\mathrm{d}^2r', \tag{3}$$

we get a new Hamiltonian

$$H = \frac{\hbar^2}{2m}\int \chi^+\left(-\mathrm{i}\nabla + \nabla\alpha - \frac{e}{c\hbar}\mathbf{A}\right)^2\chi\,\mathrm{d}^2r + \int \frac{U(|\mathbf{r}-\mathbf{r}'|)}{2}\chi^+(\mathbf{r})\chi^+(\mathbf{r}')\chi(\mathbf{r}')\chi(\mathbf{r})\,\mathrm{d}^2r\,\mathrm{d}^2r', \tag{4}$$

where $U(r)$ is the Coulomb interaction. We want to consider a set of periodic vortex-like singularities in $\nabla\alpha$. Vector $\nabla\alpha$ can be expressed in terms of the Weierstrass zeta function used in the theory of rotating superfluids [8]; it is given by the converging series

$$\zeta = \frac{1}{z} + \sum_{T_{nn'}\neq 0}\left(\frac{1}{z-T_{nn'}} + \frac{1}{T_{nn'}} + \frac{z}{T_{nn'}^2}\right), \tag{5}$$

where $z = x + \mathrm{i}y$ is a complex coordinate in the 2D plane, $T_{nn'} = n\tau + n'\tau'$ and τ, τ' are the minimal complex periods [9] of the vortex lattice. The phase factor $\mathrm{e}^{\mathrm{i}\alpha}$ will be a simple function in the 2D plane if $\nabla\alpha = K(\mathrm{Re}\,\zeta, \mathrm{Im}\,\zeta)$ and

$$\alpha(\mathbf{r}) = K\int_{\mathbf{r_0}}^{\mathbf{r}}(\mathrm{Re}\,\zeta\,\mathrm{d}x + \mathrm{Im}\,\zeta\,\mathrm{d}y), \tag{6}$$

with integer K of any sign. The quantity K and the periods τ, τ' define the topological class of the multiparticle wavefunction. The transformed Hamiltonian (4) has singularities at vortex positions and cannot be restored to the initial form (3) by any smooth finite transformation of the function α. That makes it topologically stable. We shall investigate the peculiarities of the ground state and excitations in this model at low temperatures.

1. The energy gain due to vortex formation

In this section we show that any state without a macroscopic current is unstable in an external magnetic field due to the lowering of the free energy after the formation of an isolated vortex. The change of the free energy of a charged system in a given external magnetic field is (see, e.g., [14])

$$\delta F = -\frac{1}{c}\int \mathbf{A}\delta\mathbf{j}\,\mathrm{d}V$$

where $\mathbf{A}$ is the external vector potential, $\delta\mathbf{j}$ is the change in the current and the integration is over the volume of the sample. That is also true for 2DES where the external magnetic field cannot be essentially modified by a weak 2D current. This equation can be integrated to give

$$F = E - \frac{1}{c}\int \mathbf{A}\mathbf{j}\,\mathrm{d}^2r \tag{7}$$

where E is the internal energy. The difference between the internal energy and the free energy is essential and implies the existence of vortices decreasing the free energy.

A standard assumption in the theory of 2DES in a strong magnetic field was the possibility of constructing the ground state by projection on the states in the lowest LL (e.g., [15]). In that case the average electron current vanishes at distances of the order of the magnetic length.

It is possible to calculate the change in free energy (7) due to the formation of an isolated vortex. It is convenient to use the axial gauge where the external vector potential is

$$\mathbf{A}(\mathbf{r}) = \tfrac{1}{2}rB\mathbf{e}_\phi$$

with $\mathbf{e}_\phi$ being the unit vector in the azimuthal direction. The effective vector potential is the sum of the external vector potential and the additional vector potential of the vortex

$$\delta\mathbf{A} = \frac{c\hbar}{e}\frac{K}{r}\mathbf{e}_\phi.$$

This variation of the effective vector potential corresponds to the phase α in the canonical transformation (1) with $\nabla\alpha = K/r\mathbf{e}_\phi$ which is the change due to the formation of the vortex at the origin. The corresponding electric current operator reads

$$\hat{\mathbf{j}} = \frac{\hbar e}{2Ml_B}\left\{\psi^+\left(-\frac{\mathrm{i}\partial}{\rho\partial\phi} + \frac{\rho}{2}\mathbf{e}_\phi - \frac{K}{\rho}\mathbf{e}_\phi\right)\psi + \left(\frac{\mathrm{i}\partial}{\rho\partial\phi}\psi^+ + \frac{\rho}{2}\mathbf{e}_\phi\psi^+ - \frac{K}{\rho}\mathbf{e}_\phi\psi^+\right)\psi\right\}, \tag{8}$$

where $\rho = r/l_B$. We shall measure all energies in units of $\hbar^2/(2Ml_B^2)$.

We assume the lowest LL to be partially filled; the projected form of the Fermi operators is

$$\psi = \sum_m \exp(-\mathrm{i}m\phi)R_m(\rho)c_m, \tag{9}$$

$$\psi^+ = \sum_m \exp(\mathrm{i}m\phi)R_m(\rho)c_m^+, \tag{10}$$

where c_m, c_m^+ are Fermi operators and

$$R_m(\rho) = \rho^m \exp(-\rho^2/4) N_m^{-1/2},$$

with $N_m = 2^m l_B^{2(m+1)} m!$. It is easy to show that the total azimuthal current through any ray $\phi = \text{const}$ vanishes at $\delta\mathbf{A} = 0$. The change of the second term with the magnetic moment in equation (7) reads

$$\delta F_2 = 2\pi \int_0^R \rho \, \mathrm{d}\rho \left[\rho \sum_{m>0} R_m^2 \left(-\frac{K}{\rho} \right) \langle c_m^+ c_m \rangle \right].$$

The angular brackets denote the quantum mechanical average over the assumed uniform projected ground state.

We consider large distances from the position of the vortex where the perturbation of the basic state is small and this expression can be calculated in the first order as the average over the supposed projected ground state. Thus the change δF_2 due to vortex formation is proportional to the sample area

$$\delta F_2 = -K \int_0^R n_\mathrm{e} 2\pi r \, \mathrm{d}r, \tag{11}$$

where n_e is the average electron density and the integral gives the total number of electrons in the sample. It is essential that the main contribution comes from large distances where the states are distorted quite weakly and the interaction and the microscopical structure are not changed.

The calculation of the internal energy can be done in a similar manner. The change of the internal energy due to vortex formation is given by the kinetic energy term which reads

$$E' = \int_0^R \frac{2K}{\rho} \sum_{m>0} \left(\frac{\rho}{2} - \frac{m}{\rho} \right) R_m^2 \langle c_m^+ c_m \rangle 2\pi r \, \mathrm{d}r + \int_0^R \frac{K^2}{\rho^2} \sum_{m>0} R_m^2 \langle c_m^+ c_m \rangle 2\pi r \, \mathrm{d}r. \tag{12}$$

To the same order of perturbation theory this expression has a logarithmic dependence on the sample size because the first term vanishes. The finite value of E' is obtained by making a cut-off in the vortex core at small distances where the electron density must be reduced.

Thus, vortex formation gives a gain in the energy of large enough samples when the negative magnetic moment term in the free energy exceeds the logarithmic increase of the internal energy in equation (7). We see that the supposed ground state projected on the lowest LL is unstable with respect to vortex formation. This statement is independent of the microscopic structure and of the interaction; it is valid only due to the different sample size dependences of the internal energy and the magnetic moment term of the 2DES in close analogy to the case of the rotating liquid [12].

The regularization is essential to obtain this result. There are two possibilities for regularization known from the theory of superfluid ^{3}He [16]. The simplest are singular vortices with hard cores defined by the Coulomb interaction and the atomic structure of the underlying semiconductor of the heterostructure. That gives an estimate of the order of the Bohr radius for the core size. The other possibility is a soft core with the size defined by the extension of the electron phase space like in the Skyrmion texture [17]. That gives a core of the order of the magnetic length. This case is much more complicated and we restrict our consideration to lattices of hard core vortices.

2. Magnetic translations

We assume the existence of vortex lattices that decrease the free energy according to section 1. Having in mind large magnetic fields, it is interesting to consider a simplified version of the Hamiltonian (4) without the interaction term

$$H' = \frac{\hbar^2}{2m} \int \chi^+ \left[-\mathrm{i}\nabla + \nabla\alpha - \frac{e}{c\hbar}\mathbf{A}(\mathbf{r}) \right]^2 \chi \, \mathrm{d}^2 r. \tag{13}$$

This Hamiltonian has properties close to those of a Hamiltonian with a constant magnetic field. Indeed, the translation by any period $\vec{\tau}$ of the vortex lattice gives an additional constant in the brackets:

$$\mathbf{r} \to \mathbf{r} + \vec{\tau}, \tag{14}$$

$$\left[-\mathrm{i}\nabla + \nabla\alpha - \frac{e}{c\hbar} A(\mathbf{r}) \right] \to \left[-\mathrm{i}\nabla + \nabla\alpha - \frac{e}{c\hbar}\mathbf{A}(\mathbf{r}) + \vec{\delta}(\vec{\tau}) - \frac{e}{c\hbar}\mathbf{A}(\vec{\tau}) \right], \tag{15}$$

due to the properties of Weierstrass function, $\zeta(z+\tau) = \zeta(z) + \delta(\tau)$, and the linear dependence of the external vector potential $\mathbf{A}(\mathbf{r})$ at constant magnetic field. The additional constant terms can be removed by a gauge transformation of the field operators χ, χ^+. Thus the proper magnetic translation does not change the Hamiltonian (13) and the interaction term in the full Hamiltonian (4).

If we introduce the 'effective' vector potential $\mathbf{A}_\mathrm{eff} = \mathbf{A} - \frac{c\hbar}{e}\nabla\alpha$, the magnetic translation is given by the transformation

$$T_m(\vec{\tau})\chi(\mathbf{r}) = \chi(\mathbf{r} + \vec{\tau}) \exp\left(\frac{\mathrm{i}e}{c\hbar} \mathbf{A}_\mathrm{eff}(\vec{\tau})\mathbf{r} \right), \tag{16}$$

for any real period of the vortex lattice.

It is easy to associate $\mathbf{A}_\mathrm{eff}(\vec{\tau})$ with the 'effective' magnetic flux through the unit cell of the vortex lattice given by a contour integral along its boundary

$$\Phi = \oint \mathbf{A}_\mathrm{eff} \, \mathrm{d}\mathbf{r} = \mathbf{B}\vec{\tau}_1 \times \vec{\tau}_2 + K\Phi_0 \tag{17}$$

where $\Phi_0 = 2\pi \frac{e}{c\hbar}$ is the flux quantum, and B is the external magnetic field.

As was shown by Brown [10] and Zak [11] (see also [12]), a simple finite-dimensional representation of the ray group of magnetic translations can be obtained only for a rational number of flux quanta per unit cell,

$$\Phi = \frac{l}{N}\Phi_0 = Bs + K\Phi_0, \tag{18}$$

where s is the area of a unit cell of the vortex lattice, l and N are coprime integers.

Thus the situation for vortex lattices is isomorphic to the case of a superposition of a uniform magnetic field with a rational number of flux quanta per unit cell and a periodic magnetic field with zero flux. It is therefore possible to use all the argumentation of the paper [10] and construct a finite-dimensional representation of the ray group of magnetic translations. In order to construct the representation one must impose certain boundary conditions on the solutions of the Schrödinger equation with the Hamiltonian (13). The simplest is magnetic periodicity,

$$T_m(\mathbf{L})\,\chi(\mathbf{r}) = \chi(\mathbf{r}), \tag{19}$$

where $\mathbf{L} = \mathbf{L_1}, \mathbf{L_2}$ defines the size of the sample, and $\mathbf{L_1} = NM_1\vec{\tau}_1$, $\mathbf{L_2} = NM_2\vec{\tau}_2$ with integer M_1, M_2. It is easy to show that any function χ magnetically translated according to (16) will also satisfy (19). The simplest realization is the vortex lattice consisting of exactly $N \times N$ unit cells.

These conditions are the analogues of Born–von Karman conditions in the absence of a magnetic field. Indeed, in a large enough system the density of states practically does not depend on the exact form of the boundary conditions. But the restriction to finite representations is important.

The matrices of the group representation depend on the choice of the gauge. We choose the total effective vector potential to have $A_{\mathrm{eff},y} = 0$ and assume $A_{\mathrm{eff},x} \neq 0$; the basic periods of the vortex lattice are $\vec{\tau}_1 = (\tau_x, 0)$ and $\vec{\tau}_2$. We shall use this gauge and the coordinate frame in the rest of the paper. The states of the crystal with $NM_1 \times NM_2$ unit cells of the vortex lattice can be labelled with the quasi-momentum $\mathbf{q} = q_1\mathbf{b}_1 + q_2\mathbf{b}_2$ where $\mathbf{b}_1, \mathbf{b}_2$ are the base vectors of the reciprocal vortex lattice.

The representation of the magnetic translation group is given by the matrices [10]

$$D^{\mathbf{q}}(\vec{\tau}_j) \equiv \exp(-\mathrm{i}q_j\tau_j)D(\vec{\tau}_j), \tag{20}$$

where

$$\begin{aligned} &D_{jk}(0) = \delta_{jk}, \qquad D_{jk}(\vec{\tau}_1) = \delta_{jk}\exp \mathrm{i}(j-1)\frac{l}{N}, \\ &D_{j,k}(\vec{\tau}_2) = \delta_{j,k-1}\ (\mathrm{mod}N) \qquad (j,k = 1,2,\ldots,N), \end{aligned} \tag{21}$$

and the general matrix of the representation is

$$\begin{aligned} &D_{jk}(n_1\vec{\tau}_1 + n_2\vec{\tau}_2) \\ &= \exp\left\{\mathrm{i}\pi\frac{ln_1}{N}[n_2 + 2(j-1)]\delta_{j,k-n_2}\right\}(\mathrm{mod}\,N). \end{aligned} \tag{22}$$

In general, every vector $\mathbf{q}$ corresponds to some irreducible representation of the translation group. The spacing between different vectors $\mathbf{q}$ is given by the crystal dimensions $\delta q_j = 1/NM_j$ and the domain of $\mathbf{q}$ is defined by the Brillouin zone for the given periodic part of the effective magnetic field. For the same gauge with $A_{\mathrm{eff},y} = 0$ the domain for q_1 will not change after adding a constant effective magnetic field with a nonzero flux. Therefore there are M_1N possible values of the q_1 parameter. But the domain for q_2 is reduced if one introduces a new unit cell extended in the $\vec{\tau}_2$ direction with $\mathbf{A}_2 = N\vec{\tau}_2$ in order to have an integer number of flux quanta per cell. The reduced q_2 domain is $1/N$ times smaller than in the basic reciprocal lattice. The total number of possible values of $\mathbf{q}$ for the irreducible representations in the reduced Brillouin zone will be M_1M_2N. This number must be multiplied by N due to the dimensionality of the representation, giving $M_1M_2N^2$ various states equal to the number of states in the primary Brillouin zone without a constant magnetic field. This calculation is in close analogy with the case of a zero-flux periodic magnetic field where the irreducible representations are Abelian one-dimensional representations of the translation group instead of the previously discussed N-dimensional irreducible representations. Since vectors $\mathbf{q}$ are quasi-continuous inside the reduced Brillouin zone, there are no energetic gaps within this set of states. If one takes only a part of this set, there will be no energy gaps separating it from the empty states. We suggest that the set of $M_1M_2N^2$ states is separated by some gap from a similar set with higher energies as it was for the case of a zero-flux periodic magnetic field. This must be checked numerically.

At large magnetic fields the Hamiltonian (13) will be dominating the full Hamiltonian (4) since it depends linearly on the magnetic field while the interaction term is proportional to the square root of it. In this case the energy of the ground state (with interaction taken into account) can be obtained through perturbation theory:

$$\begin{aligned} E_0 = E_0' + \tfrac{1}{2}\int U_{\mathrm{c}}(|\mathbf{r}-\mathbf{r}'|) \\ \times \langle\chi^+(\mathbf{r})\chi^+(\mathbf{r}')\chi(\mathbf{r}')\chi(\mathbf{r})\rangle\,\mathrm{d}^2r\,\mathrm{d}^2r', \end{aligned} \tag{23}$$

where E_0' is the energy of the lowest set of the states giving the irreducible representations and the angle brackets denote the average over the Slater determinant of the wavefunctions of the set (a fully filled ground state of the Hamiltonian (13)). The energy gap separating the ground state from the next set of states with higher energies at large magnetic fields must be proportional to the value of the external magnetic field. In the experiments [13], a linear dependence of the jump of the electron chemical potential in a strong magnetic field was observed for the fractions 1/3 and 2/3. The expression for the gap must be obtained by a numerical calculation of Bloch functions for a given representation and is dependent on K, N, l and the periods τ_i.

One can see that in the vortex lattice model the gap does not depend exclusively on the interaction term as was suggested in most of the theoretical works based on degeneracy of the ground Landau level. Quite the opposite: it is almost independent of the interaction in strong magnetic fields. The resolution of this paradox is the same as in the case of a rotating superfluid. The origin of the observed vortex lattices in a rotating superfluid is connected with the thermodynamic energy in the rotating frame $E' = E - \mathbf{\Omega M}$, where $\mathbf{\Omega}$ is the angular velocity and $\mathbf{M}$ is the angular momentum of the superfluid. That forces the superfluid velocity to be equal to the velocity of the solid body rotation, and the vortex lattice is a good approximation for a superfluid. In fact it is due to the different dependence of the energy on the size of the system giving preference to the solid body rotation irrespective of the microscopic internal structure of the superfluid. Section 1 confirms this analogy.

Table 1. $K = -2, l = 1$.

N	1	2	3	-5	-2	-3	-4	4	∞
ν	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{3}{7}$	$\frac{5}{9}$	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{4}{7}$	$\frac{4}{9}$	$\frac{1}{2}$

Table 2. $K = -1, l = 1$.

N	-4	4	2
ν	$\frac{4}{3}$	$\frac{4}{5}$	$\frac{2}{3}$

Table 3. $K = -1, l = 2$.

N	-7	-5	5	2
ν	$\frac{7}{5}$	$\frac{5}{3}$	$\frac{5}{7}$	$\frac{1}{2}$

The group analysis above valid for a rational number of flux quanta shows that the energy gaps are opened at special electron densities corresponding to one electron per unit cell of the vortex lattice. According to equation (18), that gives the electron density

$$n_e = \frac{B}{\Phi_0}\frac{N}{l - NK} \tag{24}$$

and must correspond to a filled set of bands obtained from $\frac{S}{s}$ states at zero average magnetic field. Here S is the sample area. A simple analysis [12] shows that this initial band is split into q subbands, each being q-fold (odd q) or $q/2$-fold (even q) degenerate, with the fraction of the number of states in each subband being equal to $1/q^2$ (odd q) or to $2/q^2$ (even q). However, the total number of states in all subbands is S/s. One can assume that these states are separated from the higher energy states by a gap also for the interacting electrons. Note that the fact of K being an even number is immaterial since the Fermi commutation rules for the operators χ and χ^+ are fulfilled automatically and have no relation to the topological number K unlike in the Jain–Chern–Simons theory. The occurrence of any specific number of vortex flux quanta can be dictated by the ground-state energy. Indeed, the experimental electron density is defined by the gate voltage or the density of the compensating charge but the observation of the specific Hall plateau can be dictated by the value of the gap at the given temperature and the purity of the sample.

The observed fractions in the FQHE correspond to those of tables 1, 2 and 3.

These fractions correspond to the celebrated Jain rule [6]. Half-filling of the Landau level $n_e = \frac{B}{2\phi_0}$ in the external field corresponds to a vanishingly small effective magnetic field (no flux quanta per unit cell).

The last two tables give the experimentally observed fractions that lie outside the Jain series. Some fractions occur more than once.

3. The half-filled Landau level

The considerations of section 1 show that vortices with the winding number $K = -1$ are thermodynamically favourable. The limiting case of a half-filled LL can be obtained either from the larger densities when $N \to (-\infty)$ or from the smaller densities when $N \to (+\infty)$. The unit cell of the vortex lattice

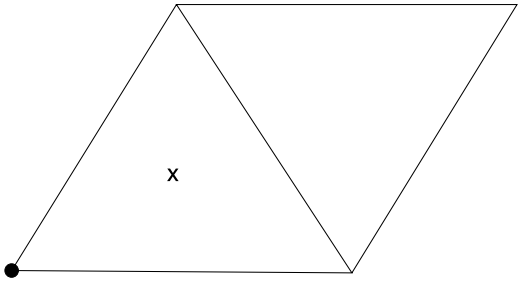

Figure 1. The vortex lattice at half-filling of the LL.

contains two vortices with $K = -1$ exactly compensating the flux of the external magnetic field. The effective magnetic field is periodic with zero flux and it is possible to use the gauge with the effective vector potential also periodic. We suppose that the vortex lattice has the form of two parallel triangular lattices for each of the vortices in the unit cell displaced by the distance $\vec{\tau}' = (\vec{\tau}_1 + \vec{\tau}_2)/3$ as shown in figure 1. The Bravais lattice corresponds to the positions of one vortex in the unit cell. The positions of the other vortices are shown by crosses.

Since the total flux is zero, one has an Abelian translation group and the electron states can be classified by their quasi-momentum. The ground state must correspond to the filling of the lowest band. The Brillouin zone for the triangle lattice has the form of a hexagon with primitive vectors of the inverse lattice being

$$\mathbf{b_1} = \frac{2\pi}{s}(\vec{\tau}_2 \times \hat{z}), \tag{25}$$

$$\mathbf{b_2} = \frac{2\pi}{s}(\hat{z} \times \vec{\tau}_1), \tag{26}$$

where s is the area of the unit cell of the vortex lattice. There are two nonequivalent vectors in the Brillouin zone, $\mathbf{q_1} = (q_x, 0)$ and the vector obtained from this one by a rotation by $\frac{2\pi}{6}$, where $q_1 = \frac{b}{\sqrt{3}}$, as shown in figure 2. The space group is isomorphic to the space group of a honeycomb 2D crystal like graphene, but the Hamiltonian corresponds to the periodic vector potential instead of the periodic potential:

$$H' = \frac{(\hbar)^2}{2M_e}\int \psi^+\left[-i\nabla - \frac{e}{c}\mathbf{A}_{\text{eff}}(\mathbf{r})\right]^2 \psi \, d^2\mathbf{r}. \tag{27}$$

The star for $\mathbf{q_1}$ consists of two rays $\mathbf{q_1}$, $\mathbf{q_2}$. A small representation of the space group corresponds to the rotations by $\pm\frac{2\pi}{3}$ giving two equivalent vectors $\mathbf{q_1} + \mathbf{b}_i$ and the reflection $y \to -y$ leaving $\mathbf{q}_1$ invariant and yielding the same lattice after a nontrivial translation that moves the crosses to points in the Bravais lattice: $(r_y|\boldsymbol{\tau}')$. We use the standard notation [18] for the elements of the space group: r_y is a reflection, τ_{r_y} is the corresponding translation. The representation of the space group can be obtained as some ray representation of the small group leaving $\mathbf{q_i}$ and the equivalent vectors invariant (see, e.g., [18]) with the matrices of the representation obeying the multiplication law

$$D(r_1)D(r_2) = \omega(r_1, r_2)D(r_1 r_2).$$

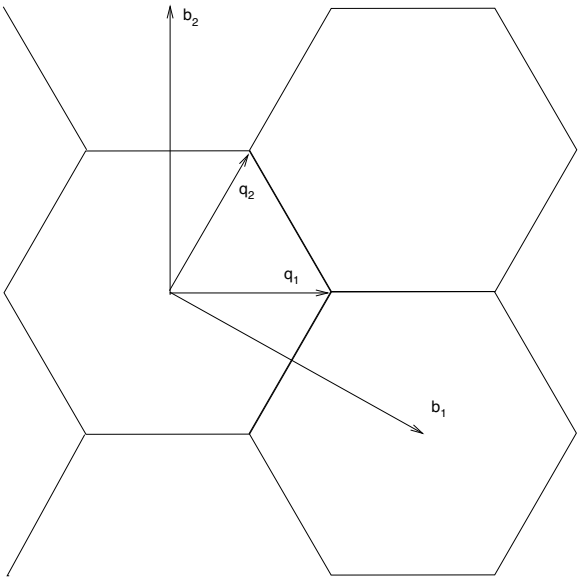

Figure 2. The reciprocal lattice and Brillouin zone at half-filling of the LL.

The representation coincides with the known representation for graphene, where a two-dimensional representation occurs at the points $\mathbf{q_1}$, $\mathbf{q_2}$ of the Brillouin zone.

We therefore have no Fermi surface, as is suggested by the CF theory, but two Fermi points at the electron density corresponding to half-filling of the LL. This result corresponds to the symmetry between the electrons and the holes. That means the absence of an energy gap and a conic Dirac spectrum

$$\epsilon_i = \epsilon^0 + |\mathbf{q} - \mathbf{q_i}|v_F,$$

in the vicinity of these points with $v_F \sim (\hbar)/(l_B M_e)$. The quantity ϵ^0 gives the electron chemical potential at half-filling.

The absence of the gap can also explain the interaction of 2D electrons with acoustic phonons, which allows us to explain the attenuation of surface waves in SAW experiments [19], like what was done in Jain–Chern–Simons theory [7].

4. The numerical calculation of the gaps

As was shown in section 2, irreducible representations of the magnetic translation group for any vortex lattice with a rational number of flux quanta are given by equation (20) with any $\mathbf{q}$ from the reduced Brillouin zone for a specific gauge $A_{\mathrm{eff},y} = 0$. The representations with different $\mathbf{q}$ are different. According to the general theorems [18], this means that the states corresponding to different representations are mutually orthogonal. The dimensionality of the representation gives N degenerate states which are also orthogonal, since the matrices of the representation are unitary. In order to perform numerical computations it is possible to use only one basic function of the representation (20).

The translation group is a subgroup of the space group of a 2D vortex lattice. The procedure for finding the irreducible representations of the space group is well known for an ordinary crystal with zero magnetic flux per unit cell. It can be generalized to the case of a rational number of flux quanta. For the simplest symmorphic case the space group is given by the product of the translation subgroup and the subgroup of the point symmetry of rotations and reflections. This gives the possibility of having some additional degeneracy due to the subgroup of the point symmetry at the specific values of the wavevector $\mathbf{q}$. For the non-symmorphic space group, there is a possibility of having representations of higher dimensionality than N at some specific values of $\mathbf{q}$. This additional degeneracy emerges if the translation subgroup does not commute with all elements of the space group at these specific values of $\mathbf{q}$. For nonspecific values of $\mathbf{q}$ the situation is the same as in the symmorphic case and one has a set of $\mathbf{q}$ with N-fold-degenerate states. We assume the symmorphic case in our computations.

Using the known irreducible representation of the translation group in the presence of the magnetic field with a rational flux (20), one can form the simplest partner function [10]

$$f_0^{\mathbf{q}} = \frac{1}{N}\sum_{n_1,n_2} \exp(\mathrm{i}\mathbf{q}_1 n_1 \vec{\tau}_1)\exp(\mathrm{i}\mathbf{q}_2 n_2 N\vec{\tau}_2) \times T_m(n_1\vec{\tau}_1)T_m(n_2 N\vec{\tau}_2)g(\mathbf{r}). \tag{28}$$

where the vectors in the reduced Brillouin zone are

$$\mathbf{q}_1 = \frac{r_1}{NM_1}\mathbf{b}_1, \qquad r_1 = 0,\ldots,NM_1 - 1$$
$$\mathbf{q}_2 = \frac{r_2}{M_2}\mathbf{b}_2, \qquad r_2 = 0,\ldots,M_2 - 1$$

and the summation is over $0 \leqslant n_1 < NM_1$, $0 \leqslant n_2 < M_2$. The other $N-1$ partner functions are given by the action of the other translations:

$$f_{m'}^{\mathbf{q}} = \exp(-\mathrm{i}m'\mathbf{q}_2\vec{\tau}_2)T_m(-m'\vec{\tau}_2)f_0^{\mathbf{q}}(\mathbf{r}) \tag{29}$$

for $m' = 1,\ldots,N-1$. These functions have the same energy and are not essential for the calculation of the energy $\epsilon(\mathbf{q})$. It is possible to calculate this energy by minimization of the mean value of the Hamiltonian (13) over the partner function (28) by specifying the unknown function $g(\mathbf{r})$. That is the standard way to use the group symmetry of the Hamiltonian. However, in this work we used a more universal and powerful method developed in [20] to obtain all eigenfunctions and eigenvalues of the representation. This method is suitable for any shape of the periodic magnetic field.

We used the regularization of the periodic part of the 'effective' vector potential $A'_{\mathrm{eff},x}(\mathbf{r})$ by Fourier truncation of the corresponding periodic magnetic field, representing the proper delta function by a finite sum $\delta(\mathbf{r}) \approx \sum_{p_x=0}^{P}\sum_{p_y=0}^{P}\cos\frac{2\pi p_x x}{L}\cos\frac{2\pi p_y y}{L}$ where $P = 10$ and L is the size of the unit cell of the vortex lattice.

It is rather onerous to perform the analysis for various vortex lattices with the same l/N flux quanta per unit cell. Instead, we have tried several specific lattices and calculated the energies for the ground band and the next by an effective numerical method and checked the existence or the absence of an energy gap between them. In these calculations we considered only the simplified model with the

Table 4. Energy gaps.

Lattice	l	K	N	ν	$\epsilon_{1\,\min}$	$\epsilon_{1\,\max}$	$\epsilon_{2\,\min}$	$\epsilon_{2\,\max}$	Δ
S	1	−2	1	1/3	0.263	0.353	0.444	0.491	0.0909
S	1	−2	2	2/5	0.269	0.293	0.392	0.405	0.0986
S	1	−2	3	3/7	0.265	0.269	0.337	0.363	0.0677
S	1	−2	−2	2/3	0.274	0.337	0.490	0.720	0.153
S	1	−1	2	2/3	0.245	0.277	0.525	0.594	0.248
T	1	−2	1	1/3	0.249	0.305	0.434	0.580	0.128
T	1	−2	2	2/5	0.227	0.234	0.360	0.384	0.125
T	1	−2	−2	2/3	0.278	0.373	0.411	0.657	0.037

Hamiltonian (13), neglecting the interaction term. We used a particular gauge where $A_{\mathrm{eff},y} = 0$ and the vortex lattice has periods $\vec{\tau}_1 = (\tau_{1,x}, \tau_{1,y} = 0)$ and $\tau_2 = (\tau_{2,x}, \tau_{2,y})$. The Schrödinger equation corresponding to Hamiltonian (13) has the form

$$\left(\tfrac{1}{2}(-\mathrm{i}\partial_x + y + A'_x)^2 + \tfrac{1}{2}(-\mathrm{i}\partial_y)^2\right)\psi = \epsilon\psi, \qquad (30)$$

where $\mathbf{A}'$ is the periodic part of the effective vector potential with zero flux, and its periods satisfy the relation $|\vec{\tau}_1 \times \vec{\tau}_2| = |2\pi l/N|$. Here the distances are measured in units of magnetic length for the constant effective magnetic field $B^0 = \frac{l}{l-NK}B$ and the energy is measured in units of the corresponding cyclotron energy $\omega_{\mathrm{c}}^0 = \omega_{\mathrm{c}}(B)|\frac{l}{l-NK}|$. The solution ψ of the equation (30) must satisfy magnetic periodic conditions (19) at the boundaries of the sample. The preliminary results of the numerical calculations are given by table 4, where by K we denote the number of flux quanta carried by a single vortex. Square (S) or triangular (T) lattices correspond to the structure formed by vortices. Energies are given in units of $\hbar eB/mc$, where B is the external uniform magnetic field. We write $\epsilon_{\min,i}$, $\epsilon_{\max,i}$ for the minimal and the maximal values of the energy $\epsilon(\mathbf{q})$ in two lowest bands $i = 1, 2$, while Δ is the energy gap.

As an example, in figure 3 we show the electron dispersion law for $l = 1$, $K = -1$, $N = 2$.

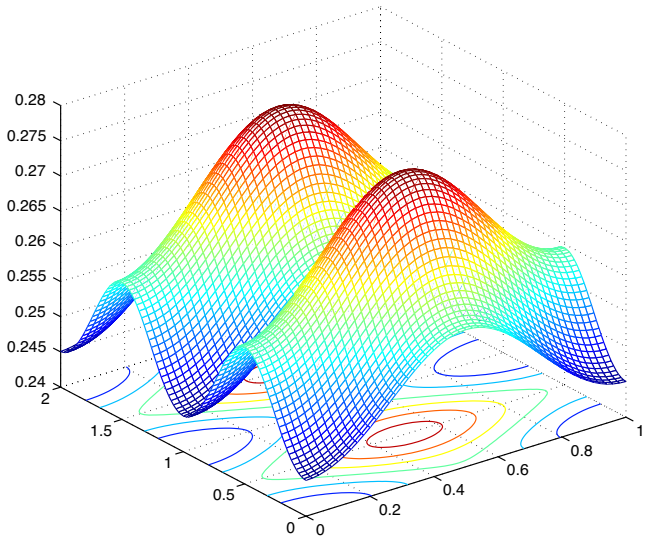

Figure 3. The dispersion law $\epsilon(\mathbf{k})$ for $l = 1$, $N = 2$, $K = -1$ and the geometry of one vortex in a square cell. The enlarged Brillouin cell corresponds to the real-lattice cell of size 2×1 in units of base periods.

(This figure is in colour only in the electronic version)

5. Conclusion

We have reproduced the key statement of the Jain's theory of composite fermions and obtained the explanation of practically all observed fractions at moderate Landau level fillings in a unified framework without any hierarchical schemes. The preliminary results were published in [21, 22]. The numerical calculations clearly show the presence of the energy gaps at experimentally observed FQHE electron densities. Our description of the states for 2DES for FQHE conditions is a kind of a mean field approximation or, more exactly, a method using a self-consistent effective vector potential. We did not consider fluctuations of the vortex field, assuming zero temperature. It is well known that in 2D the thermal fluctuations destroy the periodic order of the crystal. The same must be true for the vortex lattice. It is reasonable to suppose that nevertheless the energy gap for the charged excitations survives and the electrons form a special electron liquid which can have some kind of a plastic flow in the absence of the crystalline order. The observed Hall current may be the realization of this flow in the presence of an external electric field. The Hall constant may be determined by the mean electron density in the domain of this flow as it is in the IQHE. The domains of the electron localization due to the impurities present in the sample will induce a Hall plateau because the electron density in the flow domain is unchanged if the electron chemical potential corresponds to the energy of the localized states. The plausibility of this qualitative picture must be checked by more detailed investigations.

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